

# A note about a partial no-go theorem for quantum PCP

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## Abstract

This is not a disproof of the quantum PCP conjecture!

In this note we use perturbation on the commuting Hamiltonian problem on a graph, based on results by Bravyi and Vyalyi, to provide a partial no-go theorem for quantum PCP. Specifically, we derive an upper bound on how large the promise gap can be for the quantum PCP still to hold, as a function of the non-commuteness of the system. As the system becomes more and more commuting, the maximal promise gap shrinks.

We view these results as possibly a preliminary step towards disproving the quantum PCP conjecture posed in [AALV09]. A different way to view these results is actually as indications that a critical point exists, beyond which quantum PCP indeed holds; in any case, we hope that these results will lead to progress on this important open problem.

## 1 Introduction

The PCP theorem is arguably the most important development in computational complexity over the last two decades. In a nutshell, what it says is that given a constraint satisfaction problem (CSP), it can be efficiently replaced by one with comparable size, such that if the original one was satisfiable, the new one is too, whereas if the original one had at least one violation in each assignment, then any assignment to the new CSP must violate a *constant fraction* (!) of the constraints. Consequently, the problem of deciding whether a CSP is satisfiable or is violating a constant fraction of its constraint is NP-hard.

Is there a quantum analogue to this remarkable theorem? This is perhaps the most important open question in quantum Hamiltonian complexity, and one of the central problems in quantum complexity in general. Both a proof and disproof of this conjecture would arguably yield deep insights into the basic notions of quantum mechanics, such as entanglement, no-cloning of information, and the quantum to classical transition on large scales.

In this note we present a result that might be seen as a weak evidence *against* the existence of a quantum PCP theorem. Hopefully, it may serve as a starting point for a more general framework for disproving this conjecture, or alternatively, as a starting point for better clarifications of the conditions for a quantum PCP theorem to hold. Before stating the result, we first state what is meant by a quantum PCP theorem.

## 1.1 Background on the quantum PCP conjecture

The quantum PCP conjecture was first stated formally in Ref. [AALV09]. Here we shall roughly follow their presentation. The quantum analog of a classical CSP is the QSAT problem, which is a special instance of the local-Hamiltonian problem. In that problem, we are given a  $k$ -local Hamiltonian over a system of  $n$  qubits  $H = \sum_{i=1}^M Q_i$  that is made of  $k$ -local projections  $Q_i$  with  $M = \text{poly}(n)$ . We are promised that the ground energy of the system is either 0 (all quantum constraints are satisfied) or it is above some constant  $a = 1/\text{poly}(n)$ . Just like its classical counter-part, this problem is known to be quantum NP-complete<sup>1</sup>.

A quantum PCP theorem would state that even if  $a = rM$ , for some constant  $0 < r < 1$ , the problem is still quantum-NP hard to decide. In other words, it is quantum-NP hard to distinguish between the case when the system is completely satisfiable, or when, roughly, a fraction  $r$  of it can not be satisfied.

Formally, we may define

**Definition 1.1 (The  $r$ -gap  $k$ -QSAT problem)** *Let  $r \in (0, 1)$  be some constant. We are given a  $k$ -local Hamiltonian  $H = \sum_{i=1}^M Q_i$  over  $n$  qubits, where the  $Q_i$  are  $k$ -local projections and  $M = \text{poly}(n)$ . We are promised that the ground energy of  $H$  is either 0 (a YES instance) or is greater than  $rM$  (a NO instance). We are asked to decide which is which.*

Then the Quantum PCP conjecture is

**Conjecture 1.2 (Quantum PCP)** *There exist  $(k, r)$  for which Problem 1.1 is  $\text{QMA}_1$  hard.*

To prove that such a problem is  $\text{QMA}_1$  hard, we would like to show an efficient reduction of another  $\text{QMA}_1$ -hard problem to it. It is natural to start with the  $k$ -QSAT problem that was described above. We would like to find an efficient transformation that takes a  $k$ -QSAT problem and turns it into a  $r$ -gap  $k$ -QSAT problem such that if the original system was satisfiable (ground energy is zero), then so is the new system. On the other hand, if it was not satisfiable, with a ground energy above  $a = 1/\text{poly}$ , then the ground energy of the new system would be above  $rM$ .

This type of transformation is called “Gap Amplification”, because it amplifies the promise gap of the problem. It is precisely this type of transformation that was used iteratively in Dinur’s proof of the classical PCP theorem [Din07]. Let us describe this transformation in some more details. Dinur achieves gap amplification by an iterative process that amplifies gap by some constant factor  $> 1$  at each round. Each such iteration is made of 3 steps. One step amplifies the promise gap of the system at the expense of making it much less local. The purpose of the two other steps is to fix this, restoring the locality of the system, without compromising too much the gap amplification of the first step.

The entire process is carried over a CSP that is defined on an expander graph, and in addition, the entire proof is very combinatorial in nature. These two facts make it a promising outline for a quantum proof under the natural mapping of classical constraints to projections. Indeed, a first step in that direction was taken in Ref. [AALV09], where it was shown that essentially the amplification step in Dinur’s proof can be done also quantumly. Like the classical proof, the quantum proof relies on expander graphs. The quantization of the two other steps, however, remains an open problem.

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<sup>1</sup>It is in fact  $\text{QMA}_1$  complete and not  $\text{QMA}$  complete, which is the more natural “quantum-NP” class, but we shall not be bothered here with this technical issue. For more information on this subtle point, see Refs. [Bra06, AALV09].

## 1.2 Reasons for doubts in a quantum PCP theorem

In the attempts to prove the quantum analogue of Dinur’s proof, it seems hard to quantize any classical step that increases the size of the system; such steps seem to conflict with the quantum no-cloning principle. See Ref. [AALV09] for more details; nevertheless, such increase seems unavoidable in the classical case.

Except for these difficulties, there is another reason to believe that there is no quantum PCP theorem, pointed to us by Hastings [Has08]. As we have seen, such theorem implies the existence of systems in which it is quantum-NP hard to distinguish between a vanishing ground energy and ground energy of the order of the system size. From a physicist point of view, it is equivalent to determining whether or not the free energy of the system becomes negative at a *finite* temperature [PH10]. It is then argued that at such temperatures, on large scale, the system loses its quantum characteristics; long-range entanglement effects must fade. Consequently, the system can be described (approximately) classically, hence the problem is inside NP.

In this note, we will pursue this direction. The ultimate goal is to show that for any  $(k, r)$ , Problem 1.1 is inside NP. This, however, seems very difficult. Instead of attacking it directly, it might be beneficial to show first that a more restricted problem is inside NP. This is what we do here.

## 1.3 Results: A partial No-Go theorem for quantum PCP

We are interested in a version of problem (1.1) in which the projections are two-local, sitting on the edges of a  $D$ -regular graph:

**Definition 1.3 (The  $(d, D, r)$ -gap Hamiltonian problem on a graph)** *We consider a QSAT system  $H = \sum_{i=1}^M Q_i$  that is defined on a  $D$ -regular graph, using  $d$ -dimensional qudits that sit on its vertices, and projections  $\{Q_i\}$  that sit on its edges. We are promised that the ground energy of  $H$  is either 0 or is greater than  $rM$  for some constant  $0 < r < 1$ , and we are asked to decide which is which.*

The advantage of working with this restricted set of problems is three-fold. First, its classical analog is the outcome of Dinur’s classical PCP proof [Din07]. It is therefore a natural candidate for a quantum PCP construction. Second, it has a classical, yet non-trivial limit, which was discovered by Bravyi and Vyalı [BV03]: When the projections commute, the problem becomes classical in the sense that the ground state of the system can be described by a shallow tensor-network that can be contracted efficiently on a classical computer. This tensor-network can be given as a witness to the prover, and hence the problem is inside NP. Finally, by itself, this class of systems seems rich enough to capture non-trivial quantum effects, if exist. For example, with a  $1/\text{poly}$  promise gap, these systems become QMA<sub>1</sub>-hard [Bra06].

In this note we will go slightly beyond that classical limit. We will show that for a system which is only slightly non-commuting, Problem (1.3) is inside NP for sufficiently large  $r$ ’s. In other words, there cannot be a quantum PCP construction that yields such slightly non-commuting systems with such large  $r$ ’s.

This is the main theorem we wish to prove:

**Theorem 1.4** *For the set of QSAT systems that are defined on a  $D$ -regular graph using  $d$ -dimensional qudits, the following holds: if for every two projections,*

$$\|[Q_i, Q_j]\| \leq \delta , \quad (1)$$

*and if the system is satisfiable (has a ground energy 0), then there exists an efficiently contractable tensor network with energy  $\leq \epsilon M$ , where  $0 < \epsilon < 1$  depends only on  $d, D, \delta$ , and  $\epsilon \rightarrow 0$  as  $\delta \rightarrow 0$ . Consequently, Problem 1.3 is inside NP when its projections satisfy Eq. (1) and  $r > 2\epsilon$ .*

The idea of the proof is very simple. Using the assumption that the projections in  $H$  nearly commute, we will find an auxiliary *commuting* system,  $\hat{H} = \sum_{i=1}^M \hat{Q}_i$  such that  $\|Q_i - \hat{Q}_i\| \leq \epsilon/2$  for every  $i$ . By Ref. [BV03], this system has a ground state with an efficient description, that can thus be provided as a classical witness to the NP verifier. The point is that this state is an  $M\epsilon$  approximation of the ground state, and thus can provide an NP witness for a  $rM \geq 2\epsilon M$  approximation of the ground energy.

## 1.4 Further research

It is interesting to see if the result of this note can be strengthened. First functional dependence of  $\epsilon$  on  $d, D, \delta$  is not given here, but it is probably not too difficult to find one by generalizing the results of Ref. [PS79].

To show that Problem 1.3 is inside NP for every  $(d, D, r)$  we would like to show that for  $\delta = 1$ , one can always find a tensor-network that would yield an energy  $\leq \epsilon M$ , for an arbitrarily small, yet constant  $\epsilon$ . Such a result would be a very strong indication against quantum PCP.

There are two natural directions that may help to prove such a result. First, we note that the tensor-network that results from the construction of Bravyi and Vyalyi is a very simple one. It is essentially a depth-4 local quantum circuit. Using a classical computer, we can in fact, contract similar networks with a logarithmic depth. Can we find a perturbation theory for which the depth-4 network is just the first order approximation? Then by going to higher orders, we may systematically lower  $\epsilon$  for a given  $d, D, \delta$ .

Related to that, one may try to find a reduction of the system to a system that is more commuting, perhaps by some sort of a coarse-graining process. This again, might lead to a larger  $\epsilon$  for a given  $d, D, \delta$ . We now proceed to the proof of theorem 1.4.

## 2 Proof of theorem 1.4

### Notation:

We use the natural inner product on the space of matrices

$$\langle A|B \rangle \stackrel{\text{def}}{=} \text{Tr}(A^\dagger B) ,$$

which leads to the Frobenius norm

$$\|A\|_F \stackrel{\text{def}}{=} \sqrt{\text{Tr}(A^\dagger A)} .$$

**Proof:** Using the  $C^*$ -algebra machinery of Ref. [BV03], we start with the following decomposition of every 2-local projection in  $H$ .

**Lemma 2.1** *Let  $Q$  be a 2-local projection on  $\mathbb{C}^d \otimes \mathbb{C}^d$ . Then one can write*

$$Q = \sum_{\alpha=1}^{d^2} A_\alpha \otimes B_\alpha, \quad (2)$$

with  $A_\alpha$  and  $B_\alpha$  working locally on one particle, with the following properties:

1.  $\{A_\alpha\}$  are orthogonal and bounded  $\|A_\alpha\|_F \leq 1$ .
2.  $\{B_\alpha\}$  are orthonormal.
3. The algebra generated by  $\{A_\alpha\}$  is close under conjugation, and the same applies for  $\{B_\alpha\}$ .

**Proof:** We treat  $Q$  as a vector in a bipartite Hilbert space of operators. Using the Schmidt decomposition in that space,  $Q = \sum_\alpha \lambda_\alpha \cdot A'_\alpha \otimes B_\alpha$ , with  $\{A'_\alpha\}$  and  $\{B_\alpha\}$  orthonormal. Defining  $A_\alpha \stackrel{\text{def}}{=} \lambda_\alpha A'_\alpha$  then proves 1) and 2). The third property follows from the Hermiticity of  $Q$ . ■

The advantage of working in this representation is that a Frobenius distance between the  $Q$ 's easily translates into a Frobenius distance between the  $A_\alpha$ , and then the same applies for the usual operator norm because all norms are equivalent on finite dimensional spaces. Specifically, assume the above decomposition for adjacent projections  $Q_1, Q_2$ :

$$Q_1 = \sum_\alpha A_\alpha^{(1)} \otimes B_\alpha^{(1)} \quad Q_2 = \sum_\beta A_\beta^{(2)} \otimes B_\beta^{(2)}, \quad (3)$$

where  $A_\alpha^{(1)}$  and  $A_\beta^{(2)}$  operate on the same qudit, and  $B_\alpha^{(1)}$  and  $B_\beta^{(2)}$  on two other qudits. Then by the orthonormality of  $B_\alpha^{(i)}$ ,

$$\begin{aligned} \|[Q_1, Q_2]\|_F^2 &\stackrel{\text{def}}{=} \text{Tr} \left( [Q_1, Q_2] ([Q_1, Q_2])^\dagger \right) \\ &= \sum_{\alpha, \beta} \text{Tr} \left( [A_\alpha^{(1)}, A_\beta^{(2)}] ([A_\alpha^{(1)}, A_\beta^{(2)}])^\dagger \right) \end{aligned} \quad (4)$$

$$= \sum_{\alpha, \beta} \|[A_\alpha^{(1)}, A_\beta^{(2)}]\|_F^2. \quad (5)$$

Therefore, for every  $\alpha, \beta$ , we get  $\|[A_\alpha^{(1)}, A_\beta^{(2)}]\|_F \leq \|[Q_1, Q_2]\|_F$ .

The following lemma tells us that if the operators are slightly non-commuting we can find a set of near-by operators which fully commute.

**Lemma 2.2** *There exists a function  $\delta(\epsilon)$  with the limit  $\lim_{\epsilon \rightarrow 0} \delta(\epsilon) \rightarrow 0$  such that the following holds. Every set of operators  $\{Q_i\}$ ,  $i = 1, \dots, D$  that work on a given particle, and for which  $\|[Q_i, Q_j]\| \leq \delta(\epsilon)$ , can be replaced by the operators  $\{\hat{Q}_i\}$  with  $\|Q_i - \hat{Q}_i\| \leq \epsilon/4$ , that in addition satisfy the following properties*

1.  $[\hat{Q}_i, \hat{Q}_j] = 0$
2.  $\hat{Q}_i$  are Hermitian
3. For any other term in the system,  $\|[\hat{Q}_i, P]\|_F = \|[Q_i, P]\|_F$ , so that the system does not become less commuting at other places.

Notice that  $\delta(\epsilon)$  may depend on  $d, D$ .

**Proof:** We use Lemma 2.1 to decompose

$$Q_i = \sum_{\alpha} A_{\alpha}^{(i)} \otimes B_{\alpha}^{(i)} . \quad (6)$$

Then by what was said above, for every  $i \neq j$  and every  $\alpha, \beta$ , we have  $\|A_{\alpha}^{(i)}, A_{\beta}^{(i)}\|_F \leq \delta$ . We now claim that the operators  $\{A_{\alpha}^{(i)}\}$  can be replaced by operators  $\{a_{\alpha}^{(i)}\}$  such that  $\|A_{\alpha}^{(i)} - a_{\alpha}^{(i)}\|_F \leq c(d, D)\epsilon$ , where  $c(d, D)$  is some geometrical constant to be defined later. In addition, the operators  $\{a_{\alpha}^{(i)}\}$  enjoy the following properties

1. For every  $i \neq j$  and  $\alpha, \beta$ ,  $[a_{\alpha}^{(i)}, a_{\beta}^{(j)}] = 0$ .
2. For every  $i$ ,  $\{a_{\alpha}^{(i)}\}$  are orthogonal with  $\|a_{\alpha}^{(i)}\|_F = \|A_{\alpha}^{(i)}\|_F$ .
3. For every  $i$ , the algebra that is generated by  $\{a_{\alpha}^{(i)}\}$  is close under conjugation.

To show that these operators exist, together with the function  $\delta(\epsilon)$ , we use a neat argument from Ref. [Hal76] (page 76)<sup>2</sup>: Assume by contradiction that this is not true. Then there is a  $\epsilon_0 > 0$  and a series of operators  $\{A_{\alpha}^{(i)}(n)\}$  and  $\delta_n \rightarrow 0$  such that

$$\text{for every } i \neq j \text{ and } \alpha, \beta \quad \|[A_{\alpha}^{(i)}(n), A_{\beta}^{(j)}(n)]\|_F \leq \delta_n \rightarrow 0 , \quad (7)$$

and at the same time, for any set operators  $\{a_{\alpha}^{(i)}\}$  that fulfills the 3 requirements, for every  $n$ , there is a at least one  $a_{\alpha}^{(i)}$  such that

$$\|A_{\alpha}^{(i)}(n) - a_{\alpha}^{(i)}\| \geq c(d, D)\epsilon_0 . \quad (8)$$

But this is obviously wrong, since as we are working in a compact space (we work with bounded operators in a finite-dimensional Hilbert space), the series  $\{A_{\alpha}^{(i)}(n)\}$  must have at least one limit point. Denote this point by  $\{a_{\alpha}^{(i)}\}$ . Then it is easy to see that by Eq. (7) that property 1) must hold, and by Lemma 2.1 and the fact that they are a limit point of  $A_{\alpha}^{(i)}(n)$ , properties 2) and 3) must hold.

Having the operators  $\{a_{\alpha}^{(i)}\}$  at our hands, we construct  $\{\hat{Q}_i\}$  by simply replacing  $A_{\alpha}^{(i)} \mapsto a_{\alpha}^{(i)}$  in Eq. (6). By definition, the new operators commute between themselves, proving property 1). In addition, by using the equivalence of the operator norm and the Frobenius norm, it is an easy exercises to choose  $c(d, D)$  such that  $\|\hat{Q}_i - Q_i\| \leq \epsilon/4$ . To prove 3), let us consider  $Q \in \{Q_i\}$  and its replacement  $\hat{Q}$ , as well as a third operators  $P$  that intersects with  $Q$  but that does not belong to  $\{Q_i\}$ . We use Lemma 2.1 to write  $Q = \sum_{\alpha} A_{\alpha} \otimes B_{\alpha}$ ,  $\hat{Q} = \sum_{\alpha} a_{\alpha} \otimes B_{\alpha}$ ,  $P = \sum_{\beta} C_{\beta} \otimes D_{\beta}$ , where it is assumed that  $B_{\alpha}$  and  $C_{\beta}$  work on the same particle. Then using the orthogonality of  $A_{\alpha}, a_{\alpha}$  and  $D_{\beta}$ , we conclude that

$$\begin{aligned} \|Q, P\|_F^2 &= \sum_{\alpha, \beta} \|[B_{\alpha}, C_{\beta}]\|_F^2 \cdot \|A_{\alpha}\|_F^2 \cdot \|D_{\beta}\|_F^2 , \\ \|\hat{Q}, P\|_F^2 &= \sum_{\alpha, \beta} \|[B_{\alpha}, C_{\beta}]\|_F^2 \cdot \|a_{\alpha}\|_F^2 \cdot \|D_{\beta}\|_F^2 , \end{aligned}$$

and so 3) follows from the fact that  $\|A_{\alpha}\|_F = \|a_{\alpha}\|_F$ .

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<sup>2</sup>This nice trick was first brought to my attention by Matthew Hastings [Has09]

Finally, we show how the  $\hat{Q}_i$  can be made Hermitian. This essentially follows from the fact that for every  $i$ ,  $\{a_\alpha^{(i)}\}$  are closed under conjugation. Using this property, it is easy to see that not only  $[\hat{Q}_i, \hat{Q}_j] = 0$ , but that also  $[\hat{Q}_i^\dagger, \hat{Q}_j] = 0$ . So by replacing  $\hat{Q}_i \mapsto \frac{1}{2}(\hat{Q}_i + \hat{Q}_i^\dagger)$  we make the operators Hermitian while maintaining the other useful properties. ■

We can now finish the proof of Theorem 1.4. Using the last lemma, we construct a new 2-local Hamiltonian by sequentially going over all particles and replacing  $Q_i \mapsto \hat{Q}_i$ . We obtain a new system  $\hat{H} = \sum_i \hat{Q}_i$ , which is commuting and the distance between two corresponding operators  $Q_i$  and  $\hat{Q}_i$  is

$$\|Q_i - \hat{Q}_i\| \leq \epsilon/2. \quad (9)$$

(This is because  $Q_i$  works on 2 particles, and so it undergoes two replacements, giving us a total error of  $\epsilon/4 + \epsilon/4 = \epsilon/2$ ). Let  $|\psi_0\rangle$  be the ground state of the original system. By assumption,  $\langle\psi_0|H|\psi_0\rangle = 0$ . Then by Eq. (9),  $\langle\psi_0|\hat{H}|\psi_0\rangle \leq M\epsilon/2$ , hence the ground energy of  $\hat{H}$  is at most  $M\epsilon/2$ . By Ref. [BV03],  $\hat{H}$  has a ground state  $|\psi'_0\rangle$  that can be written as an efficient tensor-network. By using Eq. (9) once more, we see that  $\langle\psi'_0|H|\psi'_0\rangle \leq \epsilon M$ , as required. ■

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